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Quadratic Gradients on the Plane Are Generically Morse–Smale

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1. INTRODUCTION

Let $\mathscr{P}$ denote the set of all cubic polynomials in two variables with real coefficients. Each element $H \in \mathscr{P}$ has the form

$$H(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + ky + m. \quad (0)$$

By identifying each choice of coefficients for H with the corresponding element in ten dimensional Euclidean space, give $\mathscr{P}$ the *coefficient topology*. Also, define the gradient vector field of H by

$$\operatorname{grad} H = \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}\right).$$

H is a *Morse–Smale function* if (1) each critical point of grad H is hyperbolic and (2) no orbit of grad H connects two saddle points.

The purpose of this paper is to show that $\mathscr{P}$ contains an open and dense subset $\mathscr{D}$ such that every element in $\mathscr{D}$ is Morse–Smale. Using standard results (see [1] for an exhaustive treatment) each element of $\mathscr{D}$ is structurally stable.

2. QUADRATIC SYSTEMS

A first order system of autonomous differential equations on the plane

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned} \quad (1)$$

is called a quadratic system if P and Q are quadratic real polynomials. Although (1) appears simple, its subtlety is dramatically demonstrated by the intransigence ([2]) of Hilbert's Problem XVI ([5]) on the number and position of limit cycles.

A survey of qualitative results on quadratic systems (for extensive references see Coppel [4]) reveals that most of the theorems are based on several elementary facts which are collected and proved in the following theorem.

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THEOREM 1. *Let X denote the vector field of* (1) *where P and Q are relatively prime polynomials. Then*

(a) *X has at most four critical points,*

(b) *three critical points are never colinear,*

(c) *if L: $ax + by + c = 0$ is not X invariant, X has at most two critical points and contacts on L. Furthermore, if L contains two critical points and points of contact the orientation of X on the infinite segments cut off by the critical points or points of contact is opposite to the orientation of X on the finite segment.*

(d) *A line joining two critical points is an isocline.*

Proof. (a) follows from the fact that two conics meet in at most four points. If L is a line containing three critical points then L meets each conic $P(x, y) = 0$ and $Q(x, y) = 0$ in three points. This is possible only if L is a factor of both P and Q contrary to the hypothesis that P and Q are relatively prime.

Assume L is not X invariant. The critical points and points of contact of X along L are the solutions of

$$\begin{aligned} ax + by + c &= 0 \\ aP + bQ &= 0. \end{aligned} \tag{2}$$

As the solution set is the intersection of a line and a conic with no common factor there are at most two solutions. If there are two solutions the finite segment cut off on L lies in the region $aP + bQ < 0$ and the two infinite segments lie in the region $aP + bQ > 0$. This proves (c).

For (d) assume L contains two critical points and without loss of generality that one of these points lies at the origin. L is given by $y = mx$ and the slope of X along L is

$$\text{slope} = \frac{Q(x, mx)}{P(x, mx)}.$$

As $Q(x, mx)$ and $P(x, mx)$ are quadratic polynomials in one variable with equal roots their quotient is a constant. This proves (d) and completes the proof of the theorem.

When X has exactly four critical points the distribution of saddles and antisaddles is described by

THEOREM 2. *Suppose X has four critical points. If the quadrilateral with vertices at these points is convex then two opposite vertices are saddles and the other two antisaddles. If the quadrilateral is not convex then either the three exterior vertices are saddles and the interior vertex an antisaddle or the exterior vertices are antisaddles and the interior vertex a saddle.*

Proof. See Coppel [4].

3. Local Structure of Critical Points and Hyperbolicity

Since a rigid motion of the plane does not change the qualitative properties of grad H, assume that H has *standard form*

$$H(x, y) = \tfrac{1}{3}ax^3 + bx^2y + cxy^2 + \tfrac{1}{3}dy^3 + \tfrac{1}{2}(ex^2 + fy^2). \tag{3}$$

Now compute

$$\operatorname{grad} H = (ax^2 + 2bxy + cy^2 + ex,\ bx^2 + 2cxy + dy^2 + fy) \tag{4}$$

and

$$\operatorname{Hess} H = \begin{pmatrix} 2ax + 2by + e & 2bx + 2cy \\ 2bx + 2cy & 2cx + 2dy + f \end{pmatrix}. \tag{5}$$

Proposition 1. *Let $H \in \mathscr{P}$. If* grad $H(p) = 0$ *then there are two orthogonal lines L_i, $i = 1, 2$ through p such that L_i is either* grad H *invariant or p is the only point of contact along L_i. Moreover, L_i is an eigenspace of* Hess $H(p)$.

Proof. By performing a rigid motion assume $p = (0, 0)$ and H has standard form. Using (5) compute

$$\operatorname{Hess} H(0, 0) = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}.$$

The x and y axes are orthogonal eigenspaces of Hess $H(0, 0)$. Also, along the x-axis

$$\operatorname{grad} H(x, 0) = (ax^2 + ex,\ bx^2).$$

If $b = 0$ the x-axis is invariant. If $b \neq 0$ grad $H(x, 0)$ is tangent to the x-axis only at the origin. A similar argument applies along the y-axis and this completes the proof.

Recall that a critical point is ***hyperbolic*** if the Jacobian matrix of the vector field at the critical point has no eigenvalue with real part zero.

Theorem 3. *There is an open and dense subset $A \subset \mathscr{P}$ such that every element in A has only hyperbolic critical points.*

Proof. As the Jacobian matrix of grad H (= Hess H) is symmetric, a critical point fails to be hyperbolic if and only if the Jacobian vanishes. Hence, if $H \in \mathscr{P}$ has a nonhyperbolic critical point, the system

$$\begin{aligned} H_{xx}H_{yy} - H_{xy}^2 &= 0 \\ H_x &= 0 \\ H_y &= 0 \end{aligned}$$

has a real solution. Since (6) is an independent system of three algebraic equations in two unknowns it has a solution on a closed nowhere dense subset of $\mathscr{P}$. This completes the proof.

4. Saddle Connections

In order to prove that the Morse–Smale polynomials in $\mathscr{P}$ are dense, perturbations must be constructed which break saddle connections. This is easily accomplished when local perturbations are available ([7]), but since all algebraic perturbations are global new arguments must be found.

We first show that saddle connections exist. As an example consider

$$H(x, y) = x^3 + y^3 - 2xy^2 + x^2 - y^2. \tag{7}$$

The points $(0, 0)$, $(-2/3, 0)$ and $(-2, -2)$ are hyperbolic saddles for $\operatorname{grad} H$ and $(-4/5, -2/5)$ is a hyperbolic sink. As H_y vanishes on the x-axis the saddles $(0, 0)$ and $(-2/3, 0)$ are connected by a separatrix along the x-axis.

The remainder of this section is used to establish the generality of this example. We have the following remarkable fact:

Theorem 4. *If $H \in \mathscr{P}$ and* grad H *has a saddle connection then the connecting orbit lies on a straight line.*

The proof of Theorem 4 will depend on two lemmas. To facilitate the arguments to follow we fix some terminology. Let ϕ_t be the flow of $\operatorname{grad} H$. If p is a saddle point there are four separatrices with limit point p. When $\phi_t(x) \to p$ as $t \to +\infty$ the separatrix $\{\phi_t(x)\}$ is called an ω-separatrix of p and when $\phi_t(x) \to p$ as $t \to -\infty$ it is called an α-separatrix of p. Of course, if S is an α-separatrix of p and an ω-separatrix of q then S is a saddle connection from p to q. Also, the usual four quadrants at the origin determined by the coordinate axes are denoted $\underset{\sim}{Q}I$, $\underset{\sim}{Q}II$, $\underset{\sim}{Q}III$ and $\underset{\sim}{Q}IV$.

Lemma 1. *Let $H \in \mathscr{P}$ and assume*

(a) *p is a saddle point of* grad H,

(b) *S is a separatrix with limit point p,*

(c) *L is a line through p which is not invariant,*

(d) *L_+ and L_- are the two open rays of L determined by p, and*

(e) *Ω is the set of limit points of S other than p (Ω is empty or a singleton), then $(S \cap L) \cup (\Omega \cap L)$ is contained in exactly one of the rays L_+ or L_-.*

Proof. The lemma is a corollary of Proposition 1 since none of the four separatrices with limit point p intersect more than one of the four rays determined by the eigenspaces L_i, $i = 1, 2$ and the point p.

LEMMA 2. *Let $H \in \mathscr{P}$ and assume*

(a) grad H *has saddle points p and q,*

(b) *S is a separatrix from p to q, and*

(c) *S does not lie on the line through p and q,*

then there exists a third critical point r which is an antisaddle. Moreover, the line K containing p and r intersects S at a point x such that r lies between p and x along K.

Proof. By a rigid motion of the plane let p and q lie on the x-axis with p to the left of q. Theorem 1(c) states that the orientation of grad H along the x-axis must change at each critical point and nowhere else. By a time reversal, if necessary, assume grad $H(x, 0)$ has negative second component on (p, q) and positive second component on $(-\infty, p) \cup (q, \infty)$. Using Proposition 1 and the fact that S does not lie on the x-axis, the L_i at p and q are transverse to the x-axis; hence, the separatrices tangent to the orthogonal L_i are separated in pairs above and below the x-axis at the critical points p and q.

Consider first the α-separatrix S of p which is initially above the x-axis. If S does not intersect the x-axis then the region R with boundary $S \cup \{(p, q)\}$ contains the ω-separatrix at p which lies above the x-axis. Since this separatrix must then have an α-limit point r in R, r is the required critical point. If S intersects the x-axis the first point of intersection (in positive time) must occur in the segment (p, q). Let z be this intersection point and consider the region R with boundary (p, z) and that portion of S from p to z. Again the ω-separatrix above the x-axis at p is contained in R and the existence of the third critical point r follows.

If S is the α-separatrix at p initially below the x-axis suppose first that S intersects the x-axis and again let the first intersection point be z. Clearly $z \in (-\infty, p) \cup (q, \infty)$, but this violates Lemma 1. If S does not meet the x-axis, the α-separatrix at q which is initially below the x-axis lies in the region R with boundary $S \cup \{(p, q)\}$ and again the required critical point r exists.

Now in all cases, if r is a saddle point then two separatrices of r are contained in the region R. Hence, a fourth critical point s is contained in R which by Theorem 2 is necessarily an antisaddle. This completes the proof.

We now present the proof of Theorem 4.

Proof. Assume $H \in \mathscr{P}$ and H has a saddle connection S from p to q which does not lie on the line containing p and q. By Lemma 2 there is an antisaddle r such that S intersects the ray from p through r at a point z such that r is between

p and z. By a time reversal, if S traps an α-separatrix of q but not an ω-separatrix of p, assume r is a source.

By a rigid motion r lies at the origin, $p \in QII$ and H has standard form. Using what we have just concluded from Lemma 2, S must intersect QIV. This fact together with Proposition 1 applied to r implies grad $H(x, 0)$ has negative second component, grad $H(0, y)$ has positive first component and $q \in QIV$. We have

$$H(x, y) = \tfrac{1}{3}ax^3 + bx^2y + cxy^2 + \tfrac{1}{3}dy^3 + \tfrac{1}{2}(ex^2 + fy^2)$$

$$\operatorname{grad} H = (ax^2 + 2bxy + cy^2 + ex,\ bx^2 + 2cxy + dy^2 + fy)$$

with $e > 0$, $f > 0$, $b < 0$ and $c > 0$.

If $q = (u, v) \in QIV$ then $u > 0$ and $v < 0$. Assume $a > 0$ and compute $H_x(u, v)$. Clearly, $au^2 > 0$, $2buv > 0$, $cv^2 > 0$ and $eu > 0$ contrary to the fact that $H_x(u, v) = 0$. Hence, $a \leqslant 0$. Similarly, if $d < 0$, $bu^2 < 0$, $2cuv < 0$, $dv^2 < 0$ and $fv < 0$ contrary to the fact that $H_y(u, v) = 0$. Hence, $d \geqslant 0$.

For notational convenience let

$$H_x = -Ax^2 - 2Bxy + Cy^2 + Ex$$

$$H_y = -Bx^2 - 2Cxy + Dy^2 + Fy$$

where A, B, C, D, E, and F are all nonnegative and only A or D may be zero. If

$$H_x = 0$$

$$H_y = 0$$

then the resultant R vanishes. We have

$$R(x, y) = (-Ax^2 - 2Bxy + Cy^2)\,Fy - (-Bx^2 + 2Cxy + Dy^2)\,Ex$$
$$= BEx^3 - (AF + 2CE)\,x^2y - (DE + 2BF)\,xy^2 + CFy^3$$

and if (u, v) is a critical point of grad H, $R(u, v) = 0$. Clearly, $(0, 0)$ is a root of R and we must also have roots corresponding to $p \in QII$ and $q \in QIV$. Since R is a homogeneous cubic in two variables and the points p, q and $(0, 0)$ are not colinear by Theorem 1(b), R has three real homogeneous roots. Furthermore the three roots not at the origin all have nonzero ordinate. Hence, to find the homogeneous roots set $y = 1$ and note that

$$R(x, 1) = Sx^3 - Tx^2 - Ux + V$$

where S, T, U and V are all positive.

Compute,

$$\frac{dR}{dx} = 3Sx^2 - 2Tx - U$$

$$\frac{d^2R}{dx^2} = 6Sx - 2T.$$

The first derivative vanishes when

$$x = \frac{T \pm (T^2 + 3SU)^{1/2}}{3S}.$$

Hence, $R(x, 1)$ has a relative minimum with positive abscissa. Since $R(0, 1) > 0$ it follows easily that $R(x, 1)$ has one negative and two positive roots. If the homogeneous roots of R are $(-\alpha, 1)$, $(\beta, 1)$ and $(\gamma, 1)$ with α, β and γ all positive, the corresponding critical points of grad H are nonzero multiples of these roots. This implies that only one such multiple $(\lambda(-\alpha, 1))$ lies in $\underset{\sim}{Q}II \cup \underset{\sim}{Q}IV$ contradicting the assumption that $p \in \underset{\sim}{Q}II$ and $q \in \underset{\sim}{Q}IV$. This completes the proof.

5. Genericity

In this section we establish the main result.

THEOREM 5. *$\mathscr{P}$ contains an open and dense subset $\mathscr{D}$ such that each element of $\mathscr{D}$ is Morse–Smale.*

Theorem 5 now follows directly from Theorem 3 and

LEMMA 3. *Let $A \subset \mathscr{P}$ be the collection of elements in $\mathscr{P}$ with only hyperbolic critical points. If $B \subset A$ is the set of all elements with an invariant line, then B is closed and nowhere dense in $\mathscr{P}$.*

Proof. If $H \in \mathscr{P}$, compute from the general form (0)

$$H_y = 2bx^2 + 2cxy + 3\,dy^2 + fx + 2gy + k.$$

The x-axis is invariant if and only if $H_y(x, 0) = 0$ for all x. Hence, the x-axis is invariant if and only if $2bx^2 + fx + k$ vanishes identically. This implies b, f and k are all zero and a seven-dimensional subspace of the ten dimensional space $\mathscr{P}$ has the x-axis invariant. The group of rigid motions of the plane has dimension three with a one dimensional subgroup preserving the x-axis. Since every line is transformed to the x-axis by some rigid motion the subspace of $\mathscr{P}$ with invariant line has dimension nine and this completes the proof.

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